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ON GENERALIZED P.P. RINGS

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A commutative ring R is called a *generalized p.p. ring* or for short, a *g.p.p. ring* if for each a in R there exists a positive integer n (depending on a) such that a^nR is projective. In this paper we shall generalize the works of [1], [2], [4]. For instance, we prove that a commutative ring R is a g.p.p. ring if and only if R has a π-regular classical quotient ring Q and all idempotents in Q belong to R (Theorem 2); R is a g.p.p. ring if and only if R has a π-regular classical quotient ring and for each maximal ideal M of R, R_M is primary (Theorem 5). Moreover, we shall treat with formal power series rings over R and their classical quotient rings, and prove that R is a g.p.p. ring (resp. p.p. ring) if and only if some (and every) subring of the classical quotient ring of $R[[X_1, \cdots, X_m]]$ containing R is a g.p.p. ring (resp. p.p. ring) (Theorem 8).

Before stating our results we introduce the notion and terminology used in this paper. Throughout this paper R will denote a commutative ring with 1. R is called *π-regular* if for each a in R there exists a positive integer n and an element x in R such that $a^n = a^{2n}x$. By $Q(R)$ we denote the classical quotient ring of R. A ring R is said to be *quasi-regular* (resp. *quasi π-regular*) provided $Q(R)$ is regular (resp. π-regular). If K is an ideal of R, the *radical* of K, denoted by $\sqrt{K}$, consists of all elements a of R such that $a^t \in K$ for some positive integer t. Then K is called *primary* if $xy \in K$, $x \notin K$ implies $y \in \sqrt{K}$, and R is said to be *primary* if (0) is primary. By $N(R)$ we denote the prime radical of R (i.e., $N(R) = \sqrt{(0)}$), and by $E(R)$ the set of all idempotents in R. Given a subset S of the ring R, $\mathrm{ann}_R(S)$ denotes the annihilator of S in R.

We first consider the conditions for R to have a π-regular classical quotient ring.

Theorem 1. *The following are equivalent:*

1) *R is a quasi π-regular ring.*

2) *For each zero-divisor $x \in R$, there exists a positive integer n such that* $\mathrm{ann}_R(x^n) = \mathrm{ann}_R(x^{n+1})$ *and the ring* $\mathrm{ann}_R(x^n)$ *contains a non-zero-divisor.*

3) *For each $x \in R$, there exists a positive integer n and a non-zero-divisor $d \in R$ such that $x^n d = x^{2n}$.*

Proof. 1) $\Rightarrow$ 2). Let x be an arbitrary zero-divisor in R. Since $Q(R)$ is π-regular, $x^nQ(R)=x^{n+1}Q(R)$ for some positive integer n. Then $\mathrm{ann}_R(x^n)=\mathrm{ann}_{Q(R)}(x^nQ(R))\cap R=\mathrm{ann}_{Q(R)}(x^{n+1}Q(R))\cap R=\mathrm{ann}_R(x^{n+1})$. By the above, there is an element $y\in Q(R)$ such that $x^n=x^{2n}y$. Then $e=1-x^ny$ is a non-zero idempotent and $\mathrm{ann}_{Q(R)}(x^n)=eQ(R)$. Let $e=cd^{-1}$, c, $d\in R$. Then c is a non-zero-divisor of the ring $\mathrm{ann}_R(x^n)$.

2) $\Rightarrow$ 3). If x is a non-zero-divisor in R then we can take x^n as d in 3), and so we assume that x is a zero-divisor. Choose a non-zero-divisor z of $\mathrm{ann}_R(x^n)=\mathrm{ann}_R(x^{n+1})$. We shall show that x^n+z is a non-zero-divisor in R. Let $a\in\mathrm{ann}_R(x^n+z)$. Then $ax^{2n}=a(x^n+z)x^n=0$. Since $\mathrm{ann}_R(x^n)=\mathrm{ann}_R(x^{2n})$, we see that $a\in\mathrm{ann}_R(x^n)$ and hence $az=0$. But z is a non-zero-divisor of $\mathrm{ann}_R(x^n)$, and so $a=0$.

3) $\Rightarrow$ 1). Since d is invertible in $Q(R)$, it holds that $x^nQ(R)=x^{2n}Q(R)$. This implies that $Q(R)$ is π-rgular.

Next we shall generalize [2, Theorem 3.4] and [4, Theorem 1.3].

Theorem 2. *The following are equivalent*:

1) *R is a g.p.p. ring.*

2) *R is quasi π-regular and $E(Q(R))=E(R)$.*

Proof. 1) $\Rightarrow$ 2). Let x be an arbitrary zero-divisor in R. Then x^nR is projective for some positive integer n. It is easy to see that x^nR is projective if and only if $\mathrm{ann}_R(x^n)=eR$ for some $e\in E(R)$. We show $\mathrm{ann}_R(x^{n+1})$. If $a\in\mathrm{ann}_R(x^{n+1})$, then $ax\in\mathrm{ann}_R(x^n)=eR$, and so $ax=axe$. Thus $x^na=x^{n-1}xa=x^{n-1}xae=x^nea=0$. Therefore by Theorem 1 R is quasi π-regular. To prove $E(Q(R))=E(R)$, let $f\in E(Q(R))$. Then We can write $f=cd^{-1}$ for some c, $d\in R$. By hypothesis, $\mathrm{ann}_R(c^m)=gR$ for some m and for some $g\in E(R)$. Since $fQ(R)=c^kQ(R)$ for any positive integer k, we can easily see $f=1-g\in E(R)$.

2) $\Rightarrow$ 1). Let $x\in R$. Since $Q(R)$ is π-regular, there is an element $y\in Q(R)$ and a positive integer n such that $x^n=x^{2n}y$. Then by hypothesis the idempotent $e=x^ny$ is in R and hence $\mathrm{ann}_R(x^n)=\mathrm{ann}_{Q(R)}(x^n)\cap R=(1-e)Q(R)\cap R=(1-e)R$.

Corollary 3. *The following are equivalent*:

1) *R is a g.p.p. ring which contains no infinite set of orthogonal idempotents.*

2) *R is a finite direct sum of primary rings.*

We now consider the relationship between g.p.p. rings and p.p. rings. It is not difficult to see that R is a p.p. ring if and only if R is a reduced g.p.p. ring. More generally we have the following.

Proposition 4. *If R is a g.p.p. ring then $R/N(R)$ is a p.p. ring.*

Proof. Let x be an arbitrary non-nilpotent element in R. By hypothesis there exists a positive integer n and a non-zero-divisor of $(1-e)R$. Let us set $\bar{R} = R/N(R)$. We shall show that $\bar{x}^n = x^n + N(R)$ is a non-zero-divisor of $(1-\bar{e})\bar{R}$. If $d \in (1-e)R$ and $dx^n \in N(R)$, then $(dx^n)^m = 0$ for some positive integer m. Since x^n is a non-zero-divisor of $(1-e)R$, we see $d^m = 0$, that is $d \in N(R)$. Thus $\bar{x}^n$ is a non-zero-divisor of $(1-\bar{e})\bar{R}$, which implies that $\mathrm{ann}_{\bar{R}}(\bar{x}^n) = \bar{e}\bar{R}$. Since $\bar{R}$ is reduced, we can easily see that $\mathrm{ann}_{\bar{R}}(\bar{x}) = \mathrm{ann}_{\bar{R}}(\bar{x}^n)$. In consequence, we have proved $\mathrm{ann}_{\bar{R}}(\bar{x}) = \bar{e}\bar{R}$.

Remark. Suppose R is quasi π-regular. Then, using Theorem 1, we can also prove that $R/N(R)$ is quasi-regular.

The next corresponds to [1, Proposition 1].

Theorem 5. *The following are equivalent:*

1) *R is a g.p.p. ring.*

2) *R is quasi π-regular and for each maximal ideal M of R, R_M is a primary ring.*

Proof. 1) ⇒ 2). By Theorem 2, R is quasi π-regular and $E(Q(R)) = E(R)$. Let M be a maximal ideal of R, and set $K = \{a \in R \mid sa = 0 \text{ for some } s \in R-M\}$. For each $e \in E(Q(R))$ $(= E(R))$, either $e \in R-M$ or $1-e \in R-M$. Thus either $1-e \in K$ or $e \in K$. Since $Q(R)$ is π-regular, we can easily see that $KQ(R)$ is a primary ideal of $Q(R)$. Combining this with $KQ(R) \cap R = K$, we also see that K is primary. If S denotes the canonical image of $R-M$ in $\bar{R} = R/K$, each element of S is a non-zero-divisor and R_M is isomorphic to the localization of $\bar{R}$ by S. Therefore, since $\bar{R}$ is primary, R_M $(\simeq \bar{R}_S)$ is primary.

2) ⇒ 1). Let M be a maximal ideal of R, and define K in the same way as above. We show that K is a primary ideal of R. Given $a, b \in R$ such that $ab \in K$. Then, by the definition of K, we see $\bar{a}\bar{b} = 0$ in R_M. Since R_M is primary, either $\bar{a} = 0$ or $\bar{b} \in N(R_M)$, and so either $a \in K$ or $b \in \sqrt{K}$. Thus we have shown that K is primary. Since $KQ(R) \cap R = K$, we can easily see that $KQ(R)$ is a primary ideal of $Q(R)$. Therefore,

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for each $e \in E(Q(R))$, either $e \in KQ$ or $1-e \in KQ$. If $e \in KQ$, then $se=0$ for some $s \in R-M$. On the other hand, if $1-e \in KQ$ then $s'(1-e)=0$ for some $s' \in R-M$, that is, $s'e=s'$. Now we show that e is in R. Let $T=\{a \in R \mid ae \in R\}$. As we have just seen above, there is no maximal ideal which contains T. Thus $T=R$, and hence $e \in R$, proving our assertion. Therefore, by Thorem 2, R is a g.p.p. ring.

Finally, we shall investigate formal power series rings and their classical quotient rings. We begin with some preliminary results.

Lemma 6. *Let* $R((X))=\{\sum_{n=r}^{\infty} a_n X^n \mid a_n \in R,\ r \in \boldsymbol{Z}\}$. *Then it holds that* $E(R((X)))=E(R)$.

Proof. We first show that if $e=a_0+a_1X+\cdots$ is an idempotent then $e \in R$. Suppose to the contrary $e \notin R$, and let n be the smallest positive integer such that $a_n \neq 0$. Then we obtain $a_0^2=a_0$ and $a_n=2a_0a_n$. From these we see that $2a_0a_n=(2a_0)^2a_n=4a_0a_n$, and hence $a_n=2a_0a_n=0$, a contradiction.

Next we shall prove that if $e=a_mX^m+\cdots+a_0+a_1X+\cdots\ (m \leq 0)$ is an idempotent then e is in R. We proceed by induction on m. As we have done, our assertion is true for $m=0$. So we may assume that our assertion is true for $m \geq k+1$. In case $m=k$, we consider the ring $(R/(a_k))((X))$ and the canonical image $\bar{e}$ of e. Then, by induction hypothesis, we conclude that $a_i \in (a_k)$ for all $i \neq 0$. Since e is an idempotent, we get $a_k^2=0$, and hence $a_k=\sum_{i=k}^{0} a_{k-i}a_i=2a_ka_0$ and $a_0=\sum_{i=k}^{-k} a_ia_{-i}=a_0^2$. Therefore we have $a_k=2a_k$, namely $a_k=0$, and hence $a_i=0$ for all $i \neq 0$.

Lemma 7. *If R is quasi π-regular (resp. quasi-regular), then so is every intermediate ring containing $E(Q(R))$ between R and $Q(R)((X))$.*

Proof. First we show that $Q=Q(R)((X))$ is π-regular. Let P be an arbitrary proper prime ideal of Q. Then $P'=P \cap Q(R)$ is a prime ideal of $Q(R)$ and $Q(R)/P'$ is a field. Hence, $Q/P'Q \simeq (Q(R)/P')((X))$ is a field, and so P coincides with the maximal ideal $P'Q$. Thus, Q is π-regular (see, e.g., [3, Corollary 4]).

Next, let S be an intermediate ring between R and Q. Then for each $s \in S$ there exists a positive integer n and $d \in Q$ such that $s^{2n}d=s^n$. By Lemma 6, $e=s^nd \in Q(R)$. Then, s^n+1-e is a non-zero-divisor of S and $s^n(s^n+1-e)=s^{2n}$. Hence, S is quasi π-regular by Theorem 1.

We can now prove the following

Theorem 8. *Let m be a positive integer. A commutative ring R is a g.p.p. ring (resp. p.p. ring) if and only if $E(Q(R)) = E(R)$ and some (and every) intermediate ring between R and $Q(R)((X_1, \cdots, X_m))$ is a g.p.p. ring (resp. p.p. ring).*

Proof. Suppose R is a g.p.p. ring, and let S be an intermediate ring between R and $Q = Q(R)((X_1, \cdots, X_m))$. Then S is quasi π-regular by (Theorem 2 and) Lemma 7, and $E(Q(S)) = E(S)$ $(= E(R))$ by Lemma 6. Therefore S is a g.p.p. ring by Theorem 2.

Conversely, assume that a subring S of Q containing R is a g.p.p. ring. Let r be an arbitrary element of R. Then there exists a positive integer n and $e \in E(S)$ such that $\mathrm{ann}_S(r^n) = eS$. Since $e \in E(R)$ by Lemma 6, we have $\mathrm{ann}_R(r^n) = \mathrm{ann}_S(r^n) \cap R = eS \cap R = eR$. Therefore R is a g.p.p. ring.

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